FINITE VELOCITY OF RADIATIVE HEAT TRANSPORT

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The theoretical treatment of intense heat transport processes is difficult because it is necessary to take into account the variable thermal and physical properties of the medium. An example of such a process is radiative heat transport. A fundamental characteristic of the material for radiative heat transport is the path length of the radiation, which depends significantly upon temperature [1]. In [2], the process of radiative heat transport was treated using the approximation of radiative thermal conduction [1]. In this approximation the problem reduces to the analysis of a quasilinear differential equation of the parabolic type. It is found that heat can be transported as thermal waves, where the wave front delimits the cold and hot regions of the material. Physically the existence of a wave front implies a finite velocity of heat transport. The necessary conditions which the thermal conductivity of the medium must satisfy for the existence of a front are known [3, 4]. These conditions require that the thermal conductivity be degenerate on the surface of the thermal wave front.

Radiative heat transport is described in a more exact way by nonlinear integrodifferential equations, which take into account the nonlocal nature of the interaction of radiation and matter (see, e.g., [1, 5]). In many important cases one can use the gray-body approximation [1] and assume that the absorption coefficient does not depend on the spectral composition of the radiation. For the case of plane symmetry the integrodifferential equation has the following form, when written in terms of dimensionless variables [1, 5]:

$$\frac{\partial E}{\partial t} = \varkappa^2 k(T)(U - T^4), \quad U = \frac{1}{2}\int_{-1}^{1} I\, d\mu. \tag{0.1}$$

Here $T(x, t) \geq 0$ is the temperature of the material; $x \in R$ is the coordinate along which heat is transport; t, time; $E(T) \geq 0$ $[E(0) = 0]$, internal energy of the material, and is a monotonically increasing function of temperature; $U(x, t) \geq 0$, radiative energy density; $k(T)$, coefficient of absorption of radiation by the material $[0 < k(T) < \infty$ for $0 < T < \infty$, $k(0) > 0]$; $I(\mu, x, t)$, intensity of the radiation given by

$$I = \begin{cases} I_+ = \dfrac{\varkappa^2}{\mu}\displaystyle\int_{-\infty}^{x} k[T(\zeta, t)]\, T^4(\zeta, t) \exp\left[-\frac{|P(x, \zeta)|}{\mu}\right] d\zeta, & \mu > 0, \\ I_- = \dfrac{\varkappa^2}{\mu}\displaystyle\int_{\infty}^{x} k[T(\zeta, t)]\, T^4(\zeta, t) \exp\left[-\frac{|P(x, \zeta)|}{\mu}\right] d\zeta, & \mu < 0; \end{cases} \tag{0.2}$$

$$P(x, \zeta) = \varkappa^2 \int_{x}^{\zeta} k(T(\varepsilon, t))\, d\varepsilon;$$

$\mu = \cos\theta$; θ is the angle between the x axis and the (arbitrary) direction of the radiation $(0 \leq \theta \leq \pi)$ (the condition that the improper integrals in (0.2) exist limits the possible increase in the temperature T as $|x| \to \infty$ [1]); $\kappa^2 = (L/\ell)^2$; L is the characteristic length of the region heated by the radiation; ℓ is the characteristic path length of the radiation.

The limit $\kappa^2 \to \infty$ in (0.1) corresponds to the approximation of radiative thermal conductivity. In this limit (0.1) corresponds to the quasilinear heat-conduction equation, which is actually nonlinear even if $k(T) = \text{const} < \infty$. The finite velocity of heat transport is always an effect. But, as shown below, a front can arise in the process described by (0.1) if $k(T) = \infty$ on the front of the thermal wave.

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1. Simple Wave. We first consider a particular solution of the type of a simple wave. Let $T = T(\eta)$, $I = I(\mu, \eta)$, where $\eta = x - vt$, and $v \neq 0$ is an arbitrary constant. Then (0.1) reduces to

$$-v\frac{dE}{dz} = \frac{1}{2}\int_{-\infty}^{\infty} T^4 W_1(|z-\zeta|)\,d\zeta - T^4, \tag{1.1}$$

where $W_n(y) = \int_1^{\infty} e^{-y\tau}\tau^{-n}d\tau$ is the exponential integral [6]; $z = \varkappa^2\int k(T)\,d\eta$ is the optical thickness of the material [in order to shorten the notation, the argument in the integral of (1.1) and everywhere below will be omitted]. Equation (1.1) must be supplemented by boundary conditions at $z = \infty$ or $-\infty$. In order to be specific, we put

$$T = 0,\ I = 0 \text{ for } z = \infty. \tag{1.2}$$

Integrating (1.1) with (1.2), we obtain

$$vE = \frac{1}{2}\int_{-\infty}^{\infty} T^4 \operatorname{sign}(z-\zeta)\,W_2(|z-\zeta|)\,d\zeta. \tag{1.3}$$

The integral equation (1.3) is a Hammerstein equation [7]. Integrating (1.3) twice we have

$$v\int_z^{\infty} d\zeta\int_{\zeta}^{\infty} E\,d\varepsilon = \frac{1}{2}\int_{-\infty}^{\infty} T^4 \operatorname{sign}(z-\zeta)\,W_4(|z-\zeta|)\,d\zeta + \frac{1}{3}\int_z^{\infty} T^4 d\zeta.$$

The well-known relations between the exponential integrals [6] can be used to write $W_4(|z - \zeta|) = \omega(|z - \zeta|)W_2(|z - \zeta|)$, where $\omega \in [1/3, 1]$. Using the mean-value theorem and (1.3), we obtain an ordinary differential equation

$$\omega_1 v\frac{d^2\Phi}{dz^2} = v\Phi - \frac{\delta}{3}\frac{d\Phi}{dz},\quad \Phi = \int_z^{\infty} d\zeta\int_{\zeta}^{\infty} E\,d\varepsilon, \tag{1.4}$$

where $\omega_1(z) \in [1/3, 1]$, $\delta(z) \in [0, T^4(z)/E(z)]$. We will assume that the ratio $T^4(z)/E(z)$ goes to zero as $T \to 0$ and that it goes to infinity as $T \to \infty$. The solution of (1.4) can be studied qualitatively by the well-known methods of [8]. It is found that, in the limit $z \to \infty$, $T \to 0$ (1.4) is asymptotically equivalent to

$$\frac{d\Phi}{dz} = -\frac{1}{\omega_0}\Phi,\quad \omega_0 = \omega_1(\infty) = 1,$$

and hence, by applying l'Hospital's rule, we find

$$\frac{dE}{dz} = -E,\quad z \to \infty,\quad T \to 0. \tag{1.5}$$

Comparing (1.5) with (1.4) and (0.1), we find an asymptotic representation for the radiative energy density

$$U = v\int_z^{\infty} E\,dz,\quad z \to \infty.$$

Hence, the condition $U \geq 0$, $E \geq 0$ requires that $v > 0$. Physically this means that only heating waves can exist.

Asymptotic representations for $T(\eta)$ and $E(\eta)$ also follow from (1.5). In terms of the physical variables these relations are determined from the expression

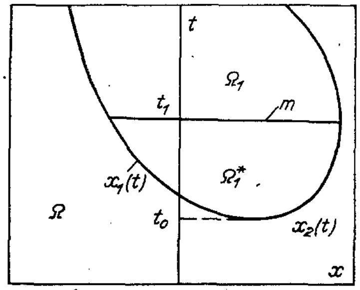

Fig. 1

$$\eta-\eta_f=-\frac{1}{\varkappa^2}\int\frac{dE}{E(T)\,k(T)}. \qquad (1.6)$$

Here η_f is an arbitrary constant, $|\eta_f| < \infty$.

It is important to note that (1.4) and the boundary condition (1.2) have the trivial solution E = 0, which is singular, since the uniqueness condition is violated when E = T = 0.

It is evident from (1.6) that if the integral

$$\int\limits_0^1\frac{dE}{E(T)\,k(T)}<\infty \qquad (1.7)$$

exists, then the boundary condition (1.2) will be satisfied for a finite value of the independent variable η. With no loss of generality we assume that the conditions (1.2) are satisfied at $\eta = \eta_f$. When $\eta > \eta_f$, the solution must be continued from the singular solution T = E = 0. Then the necessary continuity conditions for T, U, and I will be satisfied at the point $\eta = \eta_f$.

We note further that when $z \to -\infty$ (1.4) is asymptotically equivalent to

$$v\Phi-\frac{\delta}{3}\frac{d\Phi}{dz}=0,$$

and it then follows that $T(\eta)$ becomes unbounded as $z \to -\infty$ and the derivative $dT/dz \sim vE/4T^3$.

The solution of (1.1), (1.3), or (1.4) in the entire region can be found only by numerical methods. Therefore we consider an example in which the solution can be found analytically in closed form. Following [9], we put $E = T^4$; then the variables in (0.1) separate; $T(\eta)$ in this case is found by quadratures

$$\eta-\eta_f=\frac{4}{\varkappa^2\beta}\int\limits_0^T\frac{dT}{Tk(T)},$$

and $I(\mu, \eta)$ is given by

$$I=\frac{T^4}{\beta\mu+1},$$

where the separation constant $\beta \in [-1, 0]$ determines the velocity of the thermal waves

$$v=\frac{1}{\beta}\left[1-\frac{1}{2\beta}\ln\left|\frac{\beta+1}{\beta-1}\right|\right]>0.$$

The equations given here are sufficient to discuss all of the features of the solution which were addressed above.

2. Comparison Theorem. Sufficient Condition for the Existence of a Front. We proceed now to the analysis of the time evolution of an arbitrary thermal pulse specified by the initial condition $T(x, 0) = T_0(x)$. Mathematically, the problem is formulated as a Cauchy problem for (0.1) in the region $\Omega(x, t) = R_x \times R_t^+$. An effective device in the qualitative analysis of Cauchy problems is the comparison theorem with respect to the initial data. Different variants of the theorem have been proved [4, 10-13] for the approximation of radiative

heat conduction. We briefly discuss here the proof of the analogous theorem for (0.1). The proof is based on the method of [12, 13]. We omit certain nonessential details of the proof in the interest of brevity.

For the qualitative analysis of (0.1) it is convenient to transform it to the divergence form

$$\partial E/\partial t = -\partial S/\partial x, \tag{2.1}$$

where $S(x, t)$ is the radiative energy flux [5, 14]:

$$S = \frac{\varkappa^2}{2} \int_{-\infty}^{\infty} T^4 k(T) \operatorname{sign}(P) W_2(|P|)\, d\zeta. \tag{2.2}$$

In particular, the conservation of energy of the thermal pulse follows directly from (2.1):

$$\frac{d}{dt} \int_R E dx = 0$$

if $S = 0$ when $|x| = \infty$.

Let $T_i(x, t)$ $(i = 1, 2)$ be the solutions of two Cauchy problems for (2.1) given by the initial conditions $T_i(x, 0) = T_{0i}(x)$, where the functions $T_{0i}(x)$ are related by the inequality

$$T_{01}(x) \geqslant T_{02}(x),\ x \in R. \tag{2.3}$$

Following [10, 12, 13], we prove the comparison theorem by proving that its contrary is impossible. We assume the existence of the open region $\Omega_1 \subset \Omega$ and $\Omega_1 = \{(x, t): T_2 > T_1\}$, and $T_2 = T_1$, $(x, i) \in \partial\Omega_1$. We consider the region $\Omega_1{}^* \subset \Omega_1$, cut off from Ω_1 by the straight-line segment m projected to the t axis (Fig. 1). Integrating (2.1) over $\Omega_1{}^*$ and applying Green's theorem, we obtain

$$\oint_\Gamma E_i dx - \oint_\Gamma S_i dt = 0,\ \Gamma \equiv \partial\Omega_1^*,\ i = 1, 2. \tag{2.4}$$

Here it is assumed that we go around the piecewise-smooth contour Γ as indicated in Fig. 1 [the relation (2.4) can be considered as defining the generalized solution for (0.1), if the contour Γ is assumed to be arbitrary].

We consider the difference

$$\oint_\Gamma (E_1 - E_2)\, dx - \oint_\Gamma (S_1 - S_2)\, dt = 0. \tag{2.5}$$

We have the following rigorous inequality for the first integral in (2.5):

$$\oint_\Gamma (E_1 - E_2)\, dx = \int_m (E_1 - E_2)\, dx > 0, \tag{2.6}$$

because $T_1 < T_2$, $(x, t) \in \Omega_1$, and the direction of the integration is the same as the direction of circulation around the contour Γ.

The second integral in (2.5) can be transformed to

$$\oint_\Gamma (S_1 - S_2)\, dt = \frac{\varkappa^2}{2} \int_{t_0}^{t_1}\int_0^1 [(c_1 - 1)(q_{11} + q_{12}) + \Delta q_1]\, d\mu dt -$$

$$- \frac{\varkappa^2}{2} \int_{t_0}^{t_1}\int_0^1 [(c_2 - 1)(q_{21} + q_{22}) + \Delta q_2]\, d\mu dt,$$

where

$$c_i = \exp\left[-\frac{1}{\mu} |P_i(x_1, x_2)|\right];$$

$$q_{i1} = \int_{-\infty}^{x_1} T_i^4 k(T_i) \exp\left[-\frac{1}{\mu}|P_i(x_1, \zeta)|\right] d\zeta;$$

$$q_{i2} = \int_{x_2}^{\infty} T_i^4 k(T_i) \exp\left[-\frac{1}{\mu}|P_i(x_2, \zeta)|\right] d\zeta;$$

$$\Delta q_i = \int_{x_1}^{x_2} T_i^4 k(T_i) \left[\exp\left(-\frac{1}{\mu}|P_i(x_2, \zeta)|\right) + \exp\left(-\frac{1}{\mu}|P_i(x_1, \zeta)|\right)\right] d\zeta;$$

i is the number of the solution. The rest of the notation is given in Fig. 1.

If the absorption coefficient k(T) satisfies the conditions

$$k(T_1) \leqslant k(T_2),\ k(T_1)T_1^4 \geqslant k(T_2)T_2^4,\ T_1 \geqslant T_2, \tag{2.7}$$

then we have $0 < c_1(\mu, t) \leq c_2(\mu, t) < 1$, $q_{11} > q_{21}$, $\Delta q_1 < \Delta q_2$, $q_{21} > q_{22}$; hence,

$$\oint_{\Gamma} (S_1 - S_2)\, dt < 0. \tag{2.8}$$

The first of the conditions (2.7) means that the absorption coefficient does not increase when the temperature of the material increases, while the second implies the physically required increase of the luminosity with increasing temperature of the material.

Relations (2.6) and (2.8) show that (2.5) cannot be satisfied. This means that if (2.3) and (2.7) are satisfied, the solutions of the Cauchy problems are related by the inequality $T_1(x, t) \geq T_2(x, t)$ in the entire region $(x, t) \in \Omega$. This assertion is the comparison theorem with respect to the initial data.

We consider now the Cauchy problem for (0.1) with the initial condition $T(x, 0) = T_0(x)$, where $T_0(x)$ is a finite bounded function $T_0(x) > 0$ for $x \in]x_\Phi^-, x_\Phi^+[$, $x_\Phi^- < x_\Phi^+$, $|x_\Phi^\mp| < \infty$ and $T_0(x) = 0$ outside of this interval. Using the simple wave solution for (0.1) as a majorant, it follows from the comparison theorem that the relations (1.7) and (2.7) are sufficient for the existence of a surface (front of the thermal wave) $x = x_f^\mp(t)\,(x_f^-(t) < x_f^+(t)), |x_f^\mp(t)| < \infty$ such that

$$T\begin{cases} > 0, & x \in]x_f^-(t), x_f^+(t)[, \\ = 0, & x \notin]x_f^-(t), x_f^+(t)[. \end{cases}$$

The appearance of a front is due physically to the finite velocity of heat transport. The comparison theorem is not necessary for the existence of a front.

3. <u>Asymptotic Representation for the Temperature near the Front</u>. Assuming that the condition on the front $x = x_f(t)$ is satisfied asymptotically in a small neighborhood of the front, an asymptotic representation can be obtained for the solution T(x, t) in the limit $x \to x_f(t)$, and at the same time the necessary conditions for the existence of the front can be sharpened. Specifically, we will assume that $T > 0$ when $x < x_f(t)$ and $T = 0$ for $x \geq x_f(t)$.

Several methods of finding such asymptotic representations are known [12, 15, 16], and they can be used to interpret the smoothness expansion [17].

Following [15], we differentiate the condition $E(x_f(t), t) = 0$ with respect to time

$$\frac{\partial E}{\partial t} - \dot{x}_f \frac{\partial E}{\partial x} = 0,\ x = x_f(t),\ \dot{x}_f \equiv \frac{dx_f}{dt}.$$

Assuming that this equation is satisfied asymptotically when $x \to x_f(t) - 0$, and substituting for the derivative $\partial E/\partial t$ from (0.1), we obtain the relation

$$-\dot{x}_f \frac{\partial E}{\partial x} = \varkappa^2 k(T)(U - T^4),\ \dot{x}_f \neq 0,\ x \to x_f - 0. \tag{3.1}$$

Comparing (3.1) and (1.1), it can be shown that they are completely equivalent if we assume $v = \dot{x}_f$, $\eta = x - x_f(t)$. Hence, the results obtained above concerning the existence of a front for a plane wave go over completely to the case of an arbitrary moving front $x = x_f(t)$, $\dot{x}_f \neq 0$. Therefore, (1.7) is a necessary condition for the existence of a thermal wave front and a finite velocity of heat transport.

As in the case of a simple wave, (1.6) can be used to find an asymptotic representation for T close to the front. Asymptotic representations for S and U can then be found from the equations

$$S = \dot{x}_f E, \; U = \varkappa^2 \dot{x}_f \int\limits_{x-x_f}^{0} k(T) E(T)\, d\eta.$$

In particular, if $k(T) = T^{-\gamma}$, $\gamma > 0$ [1], then for an ideal gas ($E = T$), we obtain

$$T \sim [\gamma\varkappa^2(x_f - x)]^{1/\gamma}, \; S \sim \dot{x}_f T, \; U \sim \dot{x}_f T, \; x \to x_f - 0.$$

Hence if the path length of the radiation is comparable to the characteristic linear dimension of the region heated by the radiation, the nonlocal nature of the interaction of the radiation with matter is significant in the formation of a thermal wave front.

4. Approximate Analysis of the Problem. A much simpler model of radiative heat transport than (0.1) and (0.2) is the diffusion approximation. In order to obtain the fundamental equations of this approximation we integrate the definition of S in (2.2) twice:

$$\int\limits_y^\infty d\zeta \int\limits_\zeta^\infty S\, d\varepsilon = \frac{1}{2} \int\limits_{-\infty}^{\infty} T^4 \operatorname{sign}(y-\zeta) W_4(|y-\zeta|)\, d\zeta + \frac{1}{3}\int\limits_y^\infty T^4 d\zeta, \; y = \varkappa^2 \int k(T)\, dx. \tag{4.1}$$

Using (2.2) we transform (4.1) to the form

$$\int\limits_y^\infty d\zeta \int\limits_\zeta^\infty S\, d\varepsilon = \omega(y) S + \frac{1}{3}\int\limits_y^\infty T^4 d\zeta, \tag{4.2}$$

where

$$\omega(y) = \frac{\int\limits_{-\infty}^{\infty} T^4 \operatorname{sign}(y-\zeta) W_4(|y-\zeta|)\, d\zeta}{\int\limits_{-\infty}^{\infty} T^4 \operatorname{sign}(y-\zeta) W_2(|y-\zeta|)\, d\zeta}. \tag{4.3}$$

The relations between the exponential integrals [6] and the mean value theorem give $\omega(y) \in [1/3, 1]$. If we now put $\omega = \omega^* = \text{const}$, $\omega^* \in [1/3, 1]$, and differentiate (4.2), and then transform back to the physical variables, we obtain the required differential equation

$$\omega^* \frac{\partial}{\partial x} \frac{1}{\varkappa^2 k(T)} \frac{\partial S}{\partial x} = \varkappa^2 k(T) S - \frac{1}{3}\frac{\partial T^4}{\partial x}, \tag{4.4}$$

which in this approximation is postulated in place of the definition (2.2). The choice of the constant ω^* is connected physically with the method of averaging I over θ [5, 14]. As before, we have the continuity condition of the radiation [1]

$$\partial S/\partial x = \varkappa^2 k(T)(T^4 - U), \tag{4.5}$$

and it can be considered as a definition of $U(x, t)$.

Hence, in place of the integrodifferential equation (0.1), we will have the system of differential equations (2.1), (4.4), and (4.5). We note that these equations are formally equivalent to the diffusion approximation (up to the choice of the constant ω^*), which assumes the proportionality between the radiation flux S and the gradient of the radiative energy density U [1].

This approximation preserves the qualitative features of the "exact" relations (0.1) and (0.2). One can prove the comparison theorem of solutions of the Cauchy problem with respect to the initial data for the diffusion approximation in a way analogous to the method used in Sec. 2. The conditions (2.7) turn out to be necessary in this case also. The analytical and numerical study is significantly simpler in this approximation. For example, in the case of simple waves we are led at once to the ordinary differential equation

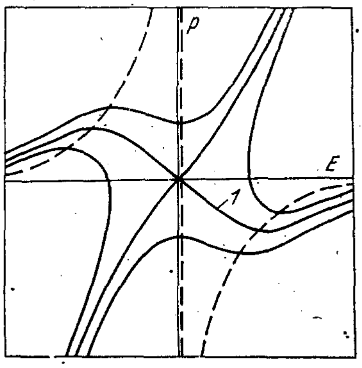

Fig. 2

$$\omega^* v \frac{d^2E}{dz^2} = vE + \frac{1}{3}\frac{dT^4}{dz}. \tag{4.6}$$

If we introduce the new dependent variable p = dE/dz, (4.6) reduces to a first-order ordinary differential equation

$$\omega^* \frac{dp}{dE} = \frac{\frac{1}{3v}\frac{dT^4}{dE}p + E}{p}, \tag{4.7}$$

where $dT^4/dE \geq 0$, $dT^4/dE \to 0$ (∞) as $T \to 0$ (∞). Obviously, the point p = E = 0 is a singular point for (4.7). The nature of the integral curves near this point can be studied by reducing (4.7) to an equivalent dynamical system [8]. This analysis shows that p = E = 0 is a saddle point. The qualitative form of the integral curves of (4.7) near the singular point is shown in Fig. 2 with v > 0. The dashed curves correspond to the extremum points of the function p(E). The condition $|dp/dE| = \infty$ is satisfied along the line p = 0. The only nontrivial solution of the problem (1.2), (4.6) is the separatrix of the family of integral curves lying entirely in the fourth quadrant of the p–E plane (denoted by the number 1).

The above qualitative analysis shows that the asymptotic representation of the solution of (4.6) in the limit $E \to 0$, $S \to 0$ can be obtained from

$$\omega^* dE/dz = -E, \ z \to \infty, \ T \to 0. \tag{4.8}$$

Relation (4.8) is the same as (1.5), except for the factor ω^*. This means that all of the asymptotic representations for $T \to 0$ obtained in Secs. 1 and 3 continue to hold here. If we put $\omega^* = 1$, they are exactly the same. The existence of the integral (1.7) leads to a thermal wave front and hence to a finite velocity of heat transport in the diffusion approximation.

We note that a numerical analysis of (4.3) can be used to refine the approximate solution obtained on the basis of (4.4).

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CONTROL OF THE SHAPE OF PHASE TRANSITION FRONTS DURING ZONE MELTING

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In order to prepare single crystals there is extensive use of the method of zone melting in which a long specimen is drawn through a heater [1, 2]. As a result of this, a molten zone occurs between the rod of polycrystalline material being consumed and the single crystal formed. Variants of the method differ in the way of heating and cooling, and also whether the specimen is contained in a crucible or not. The quality of the crystal obtained depends on the shape of the phase transition surfaces arising, which is determined by the boundary regime at the ingot surface. The problem of determining the boundary regime providing a specified (optimum in the sense of any criterion) shape of these surfaces is important. Apparently for many substances a flat shape is the optimum from the point of view of the quality of the single crystal obtained.

In this work the most simple model of the zone-melting process is considered ignoring convective heat transfer in the liquid phase. Use of this model is only valid in the case of very slow specimen movement when there is greatest interest in studying the steady-state process. It is assumed that the size of the ingot, parameters of the remelted substance, drawing rate, width of the liquid zone, and heating schedule are known. The cooling schedule is sought which provides a flat shape for the melting and crystallization fronts.

1. Statement of the Problem. Let an ingot be drawn through a heater at constant velocity v. We choose a Cartesian coordinate system (x_1, x_2, x_3) connected with the heater so that axis x_1 coincides with the direction of specimen movement. It is assumed that heat exchange is known at the boundary of the region G' which does not move in this coordinate system. The temperature field $T'(x_1, x_2, x_3)$, which is steady in the selected system, and the position of the melting Σ_1 and crystallization Σ_2 fronts are determined from the conditions

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